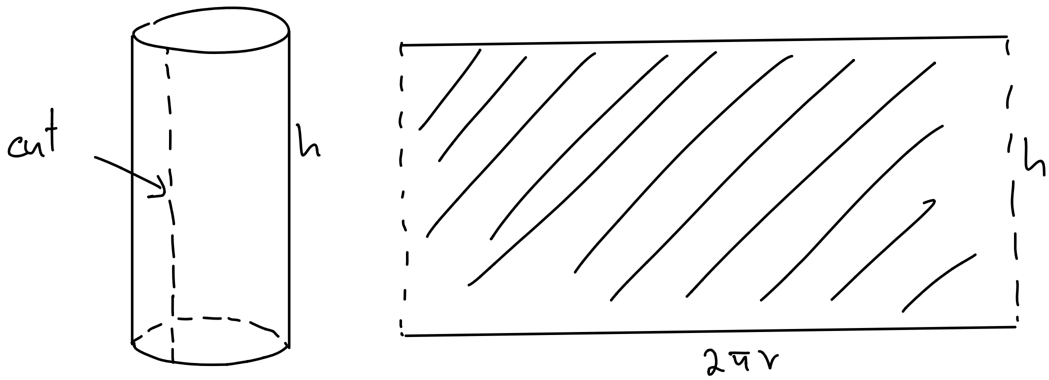
§ 8.2 Area of a Surface of Revolution

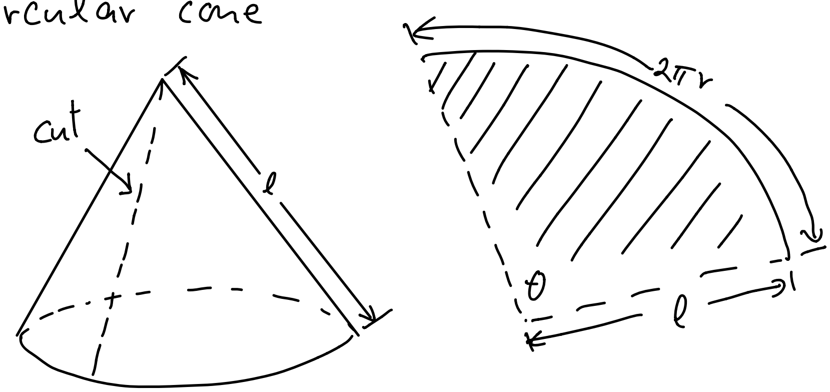
A surface of revolution is formed when a curve is rotated about a line.

- cylinder:



$$\Rightarrow A = 2\pi r h$$

- circular cone



$$\theta = 2\pi r/\ell \Rightarrow A = \frac{1}{2}\ell^2\theta = \frac{1}{2}\ell^2\left(\frac{2\pi r}{\ell}\right) = \pi r \ell$$

- band

$$A = \pi r_2(\ell_1 + \ell) - \pi r_1 \ell_1$$

$$= \pi\left[(r_2 - r_1)\ell_1 + r_2 \ell\right] \qquad (1)$$

From similar triangles we have

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \quad \text{or} \quad (r_2 - r_1) l_1 = r_1 l$$

Putting this into the first equation, we get

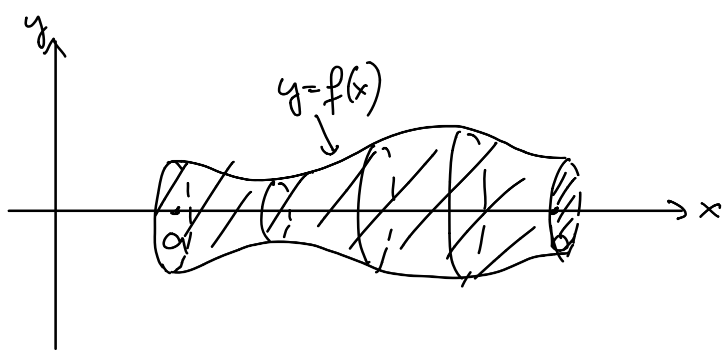
$$A = \pi (r_1 l + r_2 l) \qquad (2)$$

or $A = 2\pi r l$ where

$$r = \frac{1}{2}(r_1 + r_2)$$

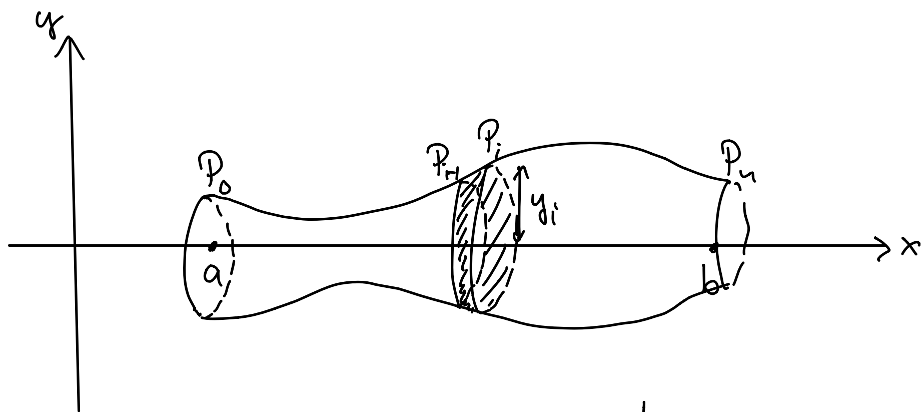
is the average radius of the band.

- general situation:



consider the above surface obtained from rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis, where f is positive and has a continuous derivative.

We devide the interval $[a, b]$ into n subintervals with endpoints $x_0, x_1, \dots, x_n$ and equal width Δx:



By formula (2) we get for the surface area of each band:

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1} P_i|$$

For $|P_{i-1} P_i|$ we get from the arc length

$$|P_{i-1} P_i| = \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

where x_i^* is some number in $[x_{i-1}, x_i]$.

When Δx is small, we have $y_i = f(x_i) \approx f(x_i^*)$ and also $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$, since f is continuous. Therefore

$$2\pi \frac{y_{i-1} + y_i}{2} |P_{i-1} P_i| \sim 2\pi f(x_i^*) \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$$

$$\Rightarrow \quad A \approx \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1+[f'(x_i^*)]^2}\, \Delta x \qquad (3)$$

Taking the limit $n \to \infty$ we obtain:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^*) \sqrt{1+[f'(x_i^*)]^2}\, \Delta x$$

$$= \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

<u>Definition 8.2</u> (surface area):

Let $f: [a,b] \to \mathbb{R}_+$ be a positive function with continuous derivative. Then we define the "surface area" obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis as

$$S = \int_a^b 2\pi f(x) \sqrt{1+[f'(x)]^2}\, dx$$

$$= \int_a^b 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad (4)$$

If the curve is described as $x = g(y)$, $c \le y \le d$, then the formula becomes

$$S = \int_c^d 2\pi y \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy \qquad (5)$$

Symbolically, we can also write this using the notation for arc length as

$$S = \int 2\pi y \, ds \quad \text{or} \quad S = \int 2\pi x \, ds$$

(for rotation about y-axis)

Example 8.4:

The curve $y = \sqrt{4-x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the surface area after rotation about the x-axis.

Solution:

$$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{4-x^2}}$$

and so, by formula (5), the surface area is

$$S = \int_{-1}^{1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \sqrt{1 + \frac{x^2}{4-x^2}} \, dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4-x^2} \, \frac{2}{\sqrt{4-x^2}} \, dx$$

$$= 4\pi \int_{-1}^{1} 1 \, dx = 4\pi(2) = 8\pi.$$

Example 8.5:

The arc of the parabola $y=x^2$ from $(1,1)$ to $(2,4)$ is rotated about the y-axis. Find the area of the resulting surface.

Solution 1:

Using $y=x^2$ and $\frac{dy}{dx}=2x$

we have,

$$S=\int 2\pi x\, ds$$

$$=\int_1^2 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$=2\pi\int_1^2 x\sqrt{1+4x^2}\, dx$$

Substituting $u=1+4x^2$, we have $du=8x\,dx$.

$$\Rightarrow\quad S=\frac{\pi}{4}\int_5^{17}\sqrt{u}\, du=\frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_5^{17}$$

$$=\frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right)$$

Solution 2:

Using $x = \sqrt{y}$ and $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

we have

$$S = \int 2\pi x \, dx = \int_1^4 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= 2\pi \int_1^4 \sqrt{y} \sqrt{1 + \frac{1}{4y}} \, dy = \pi \int_1^4 \sqrt{4y+1} \, dy$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du \qquad (\text{where } u = 1 + 4y)$$

$$= \frac{\pi}{6} \left(17\sqrt{17} - 5\sqrt{5} \right)$$

Example 8.6:

Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x < 1$, about the x-axis.

Solution:

Using formula (5) with

$$y = e^x \quad \text{and} \quad \frac{dy}{dx} = e^x$$

we have

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

$$= 2\pi \int_1^e \sqrt{1 + u^2}\, du \quad (u = e^x)$$

$$= 2\pi \int_{\pi/4}^{\alpha} \sec^3\theta\, d\theta \quad (u = \tan\theta \text{ and } \alpha = \tan^{-1} e)$$

$$= 2\pi \cdot \frac{1}{2} \left[\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| \right]_{\pi/4}^{\alpha}$$

$$= \pi \left[\sec\alpha \tan\alpha + \ln(\sec\alpha + \tan\alpha) - \sqrt{2} - \ln(\sqrt{2} + 1) \right]$$

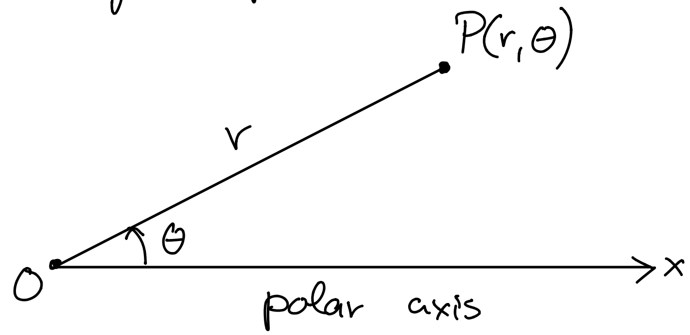
Since $\tan\alpha = e$, we have

$\sec^2\alpha = 1 + \tan^2\alpha = 1 + e^2$ and

$$S = \pi \left[e\sqrt{1 + e^2} + \ln\left(e + \sqrt{1 + e^2}\right) - \sqrt{2} - \ln(\sqrt{2} + 1) \right]$$

§ 8.3 Polar coordinates

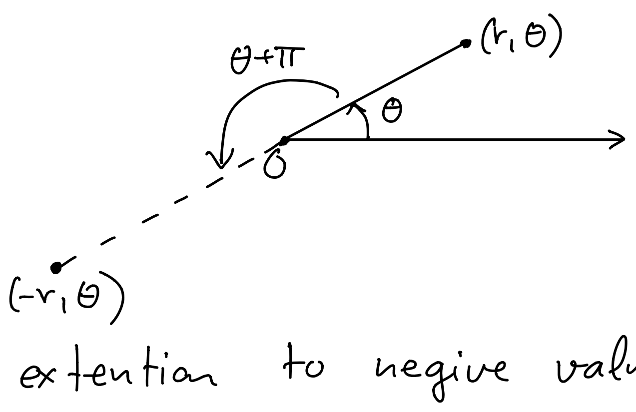
Here we describe a coordinate system introduced by Newton, called the "polar coordinate system", which is more convenient for many purposes.



The pair (r,θ) are called "polar coordinates" of P.

<u>Convention</u>:

An angle is positive if measured in the counterclockwise direction and negative otherwise.



extention to negive values of r

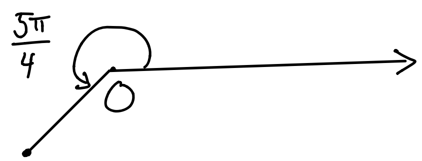
$(-r,\theta)$ represents the same point as $(r, \theta+\pi)$.

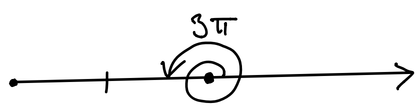
Example 8.7:

Plot the points whose polar coordinates are

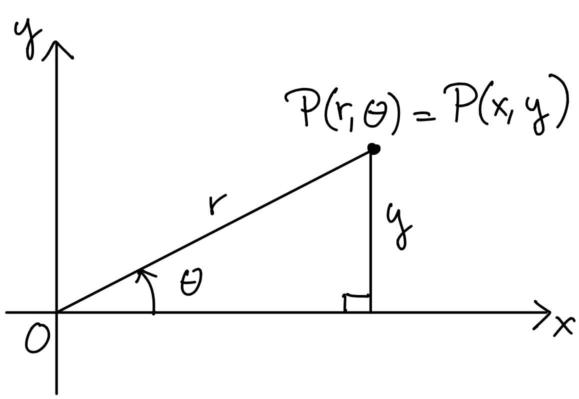
i) $(1, 5\pi/4)$ ii) $(2, 3\pi)$

solution:

i) 

ii) 

Connection between polar and Cartesian coordinates ($x-y$ coordinates):



$$\cos\theta = \frac{x}{r}$$

$$\sin\theta = \frac{y}{r}$$

$$\Rightarrow \quad x = r\cos\theta \qquad y = r\sin\theta \qquad (1)$$

These equations are valid for <u>all</u> values of r and θ.

The opposite direction is given by:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad (2)$$

Example 8.8:

i) Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

Equations (1) give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

ii) Represent the point with Cartesian coordinates $(1,-1)$ in terms of polar coordinates.

Equations (2) give

$$r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

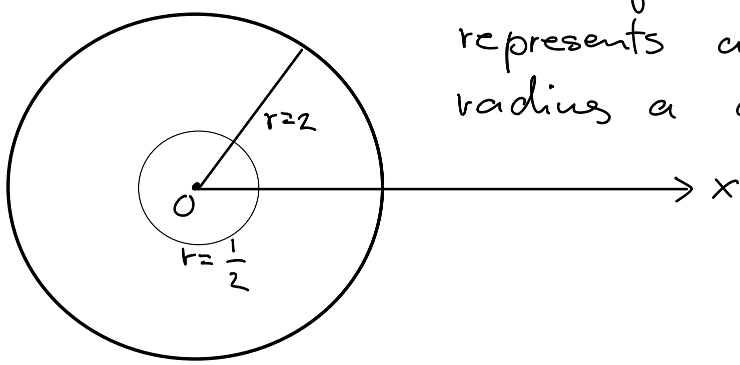
Definition 8.3:

The "graph of a polar equation" (polar curve) $r = f(\theta)$, or more generally $F(r,\theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

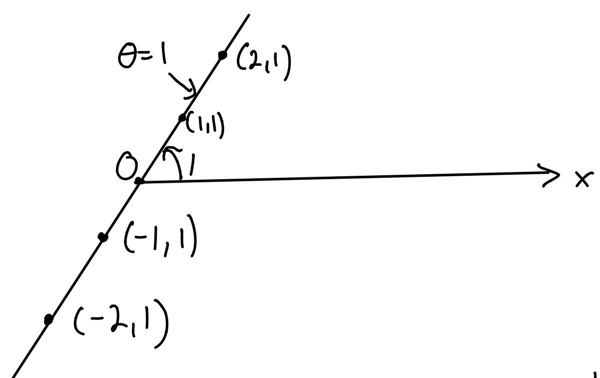
<u>Example 8.9</u> :

i) What curve is represented by the polar equation $r = 2$?

More generally, $r = a$ represents a circle with radius a and origin O.

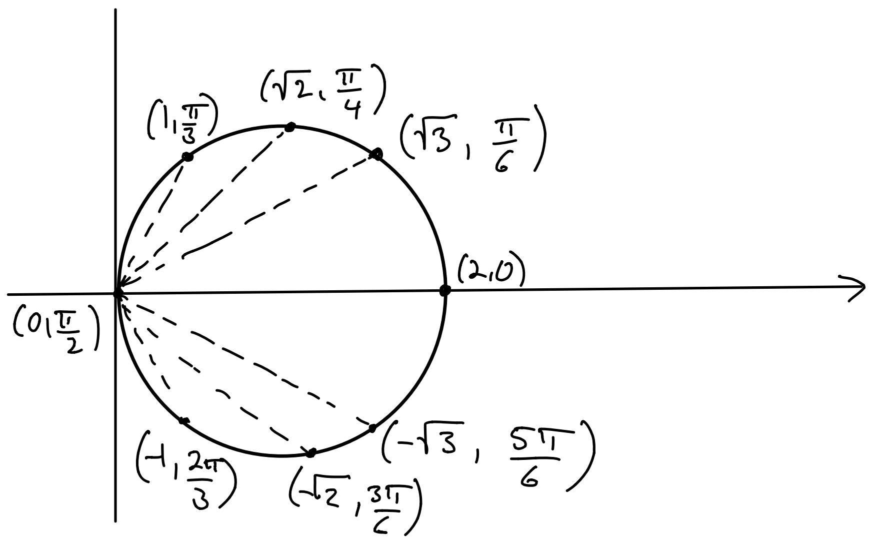


ii) Sketch the curve $\theta = 1$



iii) Sketch the curve with polar equation $r = 2\cos\theta$:

We only need values of θ between 0 and π (cos is periodic beyond π)



We can also convert the equation to a Cartesia equation and obtain:

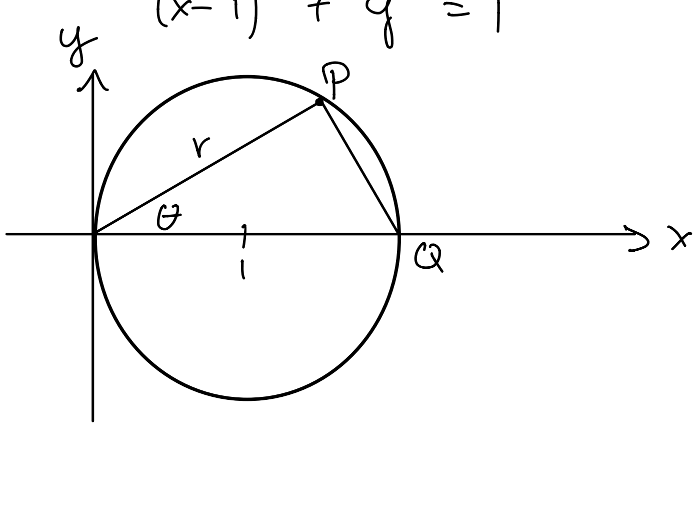
$$r = 2\cos\theta = 2x/r \qquad \left(\cos\theta = \frac{x}{r}\right)$$

$$\Rightarrow \quad 2x = r^2 = x^2 + y^2$$

$$\text{or} \quad x^2 + y^2 - 2x = 0$$

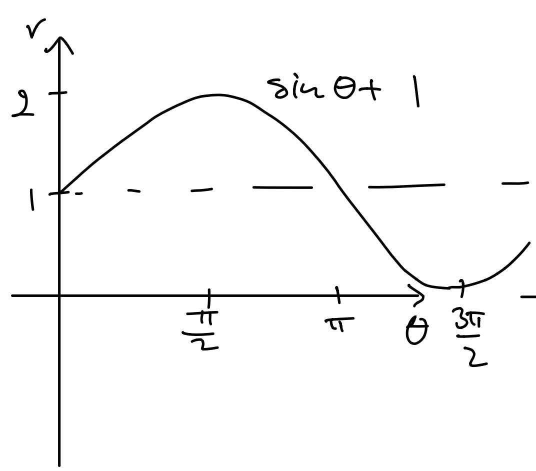
Completing the square, we obtain

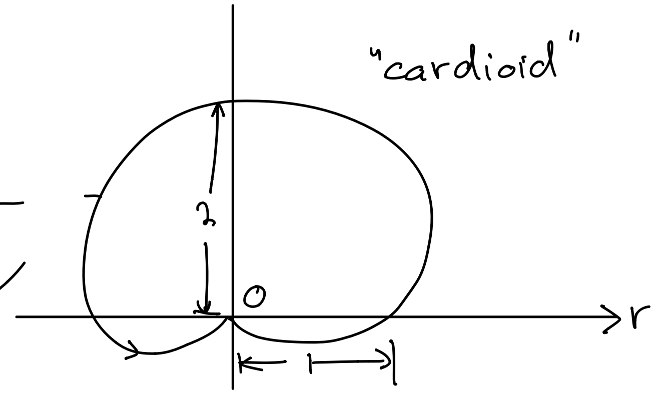
$$(x-1)^2 + y^2 = 1$$



circle with center $(1,0)$ and radius 1.

iv) Sketch the curve $r = 1 + \sin\theta$.





v) sketch the curve $r = \cos 2\theta$